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APPLICATION OF GENERAL CONSTITUTIVE PRINCIPLES TO THE DERIVATION OF MULTIDIMENSIONAL TWO-PHASE FLOW EQUATIONS

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Abstract—The process of determining appropriate constitutive equations for multidimensional time averaged two-phase flow equations is studied from the point of view of starting from general principles, and proceeding to specific constitutive equations which contain known physical effects. Energetic effects and phase change are not considered. Models are given for the interfacial momentum transfer, the laminar and turbulent (Reynolds) stresses, and the pressure differences between the phases, and between a given phase pressure and the interfacial average pressure.

1. INTRODUCTION

Predicting any features of two-phase flows is difficult, but attempting to predict multidimensional flows is particularly complicated due to the added difficulties associated with obtaining adequate models for the turbulent and interfacial transfer effects.

There are several works in the literature which use multidimensional models for two-phase flow. The book of Soo (1967), and papers by Anderson & Jackson (1967), Murray (1965) and Hinze (1972) give equations for applications to particle-fluid mixtures. Solbrig & Hughes (1978), Crowe (1975), van Wijngaarden 1968 and Drew (1971) give applications to gas–liquid systems. Of these, only Drew (1971) uses general constitutive principles.

In this paper, we shall approach the problem of determining constitutive equations for multidimensional two-phase flows starting from general considerations, and proceeding to equations which contain known physical effects for dispersed flows; that is, flows where one phase can be considered to be discrete and the other continuous (e.g. droplet or bubbly flows).

The flow situation of primary interest is the flow of steam and water in nuclear reactor components. However, the flow of air (or other gases) and water may also be of importance in nuclear reactor technology. We shall base our development on the assumption that the flow consists of two fluids, which are adequately described by the compressible Navier–Stokes fluid dynamical equations. Separating these two fluids is an interface, at which interfacial transfer is presumed to occur.

We shall not consider energetics in this paper. Our interest here is in the mechanical effects in the two-phase flows. Therefore, we shall treat the equations for conservation of mass and momentum for each phase, with the assumption that no change of phase occurs. The assumption that no phase change occurs is a limiting one; however our focus in this paper is primarily on mechanical effects, as opposed to thermodynamic effects.

2. EQUATIONS OF MOTION AND JUMP CONDITIONS

We consider a two-phase flow which consists of two fluids separated by an interface. Within the range of applications we anticipate, these fluids are compressible Newtonian fluids. It also suffices to treat the interface as an entity which can support surface tension, but has no other

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mechanical effect. For given initial and boundary conditions, the problem is uniquely specified by the Navier–Stokes equations and the jump conditions at the interface.

The Navier–Stokes equations, governing the motion of each fluid away from the interfaces, are parabolic. Therefore, if we consider the stress on the liquid side of the interface, at some interfacial position $\underline{x}_i$, the value of the stress vector is†

$$-\underline{n}_k p_k + 2\mu_k \underline{n}_k \cdot [\nabla \underline{v}_k + (\nabla \underline{v}_k)^T].$$

Both p_k and $\underline{v}_k$ are the solutions of the Navier–Stokes equations, subject to conditions at the interface and boundaries, and the appropriate initial conditions. Thus this stress depends on the flow conditions at each point in the fluid, at the present time, and during the history of the motion. However, we wish to model it in terms of the averaged flow variables. It is by no means clear that such an approach can succeed.

Since the values of all the exact field variables are uniquely determined by the initial and boundary conditions, the best resolution of the problem of predicting the two-phase flow would be to solve the equations of motion. Unfortunately, the present state of knowledge for solution methods for nonlinear partial differential equations precludes the possibility of anything more than a crude approximation to the solution in any realistic complicated flow situation.

Therefore workers in the field of two-phase flow have developed averaging techniques which are applied to the equations for two-phase flows (e.g. Ishii 1975). The philosophy behind averaging is that the exact equations contain details of the flow which are of no use on the scale of interest for the motions. Averaging the equations gives a set of "filtered" equations which does not contain the unwanted details of the flow. The price paid for the lack of unwanted detail in the averaged equations is that several terms appear in the averaged equations which are not determined by the averaging process. These terms contain the effects of the lost information, and must be determined through appropriate constitutive equations.

The average which Ishii (1975) uses is the time average defined by

$$\bar{F}(x, t) = \frac{1}{\Delta t} \int_t^{t+\Delta t} F(\underline{x}, t')\, dt', \qquad [1]$$

where the integration is taken to mean Riemannian integration. From this averaging process, phasic averaged variables and interfacial averaged variables are defined in quite natural ways.

In the following discussion, Ishii's notation and equations are used.

(1) *Conservation of mass for phase* k

$$\frac{\partial \alpha_k \bar{\bar{\rho}}_k}{\partial t} + \nabla \cdot (\alpha_k \bar{\bar{\rho}}_k \underline{v}_k) = \Gamma_k, \qquad [2]$$

where α_k is the fraction of phase k, $\bar{\bar{\rho}}_k$ is the average density of phase k, $\bar{\underline{v}}_k$ is the mass-averaged velocity of phase k, and Γ_k is the rate of mass generation of phase k at the interface.

(2) *Conservation of linear momentum for phase* k

$$\frac{\partial}{\partial t}(\alpha_k \bar{\bar{\rho}}_k \bar{\underline{v}}_k) + \nabla \cdot (\alpha_k \bar{\bar{\rho}}_k \bar{\underline{v}}_k \bar{\underline{v}}_k) = -\nabla \alpha_k \bar{\bar{p}}_k + \nabla \cdot [\alpha_k (\bar{\bar{\underline{\tau}}}_k + \underline{\tau}_k{}^T)] + \alpha_k \bar{\bar{\rho}}_k \bar{\underline{g}}_k + \underline{M}_k, \qquad [3]$$

where $\bar{\bar{p}}_k$ is the average pressure of phase k, $\bar{\bar{\underline{\tau}}}_k$ is the viscous stress tensor of phase k, $\underline{\tau}_k{}^T$ is the turbulent stress tensor, $\bar{\underline{g}}_k$ is the average acceleration, and $\underline{M}_k$ is the rate of momentum generation of phase k at the interface.

†T represents the transpose.

(3) *Conservation of energy for phase* k

$$\frac{\partial}{\partial t}\left[\alpha_k\bar{\bar{\rho}}_k\left(\bar{\epsilon}_k+\frac{\bar{v}_k^2}{2}\right)\right]+\nabla\cdot\left[\alpha_k\bar{\bar{\rho}}_k\left(\bar{\epsilon}_k+\frac{\bar{v}_k^2}{2}\right)\bar{v}_k\right]=-\nabla\cdot[\alpha_k(\bar{\bar{q}}_k+q_k^T)]$$
$$=-\nabla\cdot(\alpha_k\bar{\bar{p}}_k\bar{v}_k)$$
$$+\nabla\cdot[\alpha_k(\bar{\bar{\tau}}_k+\tau_k^T)\cdot\bar{v}_k]+E_k, \qquad [4]$$

where $\bar{\epsilon}_k$ is the specific energy of phase k, $\bar{\bar{q}}_k$ is the conductive heat flux in phase k, q_k^T is the turbulent heat flux, and E_k is the rate of generation of energy to phase k across the interface. The appropriate jump conditions are:

(1) *Interfacial mass conservation*

$$\sum_{k=1}^{2}\Gamma_k=0. \qquad [5]$$

(2) *Interfacial momentum conservation*

$$\sum_{k=1}^{2}M_k=M_m, \qquad [6]$$

where M_m represents the mixture volumetric momentum source, which results from surface tension, and depends on the geometric state of the interface.

(3) *Interfacial energy conservation*

$$\sum_{k=1}^{2}E_k=E_m, \qquad [7]$$

where E_m represents the surface energy source due to surface tension effects.

The discussion will concentrate on the mechanical aspects of the problem, and leave energetic considerations for the future. Assume that the densities $\bar{\bar{\rho}}_k$ are known, and that Γ_1 $(=-\Gamma_2)$ is zero.

The terms for which the constitutive equations must be determined include M_k, $\bar{\bar{\tau}}_k$, τ_k^T, M_m and $\bar{\bar{p}}_1-\bar{\bar{p}}_2$.

Not all these variables are independent. By [6],

$$M_1=-M_2+M_m. \qquad [8]$$

Thus, if M_m is known, we need only determine a constitutive equation for M_2, since M_1 is determined by [8].

It will be useful to examine the definition of each of the terms which must be constituted. This gives an idea of the origin of the terms to be constituted in terms of the averaging process and the exact fields involved. We have

$$M_k=\frac{1}{\Delta t}\sum_j\frac{1}{v_{n_j}}n_k\cdot T_k \qquad [9]$$

$$\tau_k^T=-\overline{\overline{\rho_k v_k' v_k'}}. \qquad [10]$$

Here, T_k is the instantaneous stress tensor, which is equal to $-p_kI+2\mu_k[\nabla v_k+(\nabla v_k)^T]$, where T denotes the transpose of the tensor.

There is one conceptual difficulty with Ishii's derivation. If the interface is motionless at $\underset{\sim}{x}$ at any time between t and $t+\Delta t$, then $v_{n_i}=0$, and $\underset{\sim}{M}_k$ is singular. It has become acceptable to treat this fundamental shortcoming of Ishii's model as an annoyance, but to proceed to use it anyway. The seriousness of the problem can be illustrated by the following argument. If the flow is bubbly, the interface does not stand still very often. If the interface happens to come to rest at $\underset{\sim}{x}_1$, at time t, then $\underset{\sim}{M}_k$ at $\underset{\sim}{x}_1$ is singular for all times t such that $t_1-\Delta t\le t\le t_1$. To some degree, then, the singularity is localized in time. Furthermore, in bubbly flow, the interface coming to rest at $\underset{\sim}{x}_1$ at t_1 is a random event, and it is unlikely that it happens again near $\underset{\sim}{x}_1$, or very soon after t_1.

In contrast, consider separated flow. If the interface is exactly at rest for all t, then $\underset{\sim}{M}_k$ is singular for all t, at any $\underset{\sim}{x}$ on the interface. If the flow is separated, but the interface is wavy, then the singularity of $\underset{\sim}{M}_k$ is spread out in space, over all points where the interface comes to rest. There are many of these points, and the occurrence of a singularity repeats in time, as the interface moves to that point, stops, and retreats. For points far from the interface, M_k is zero, since no interfacial crossings occur there.

For a well mixed flow, where the coming to rest of an interface is random and not too frequent, it is considered to be permissible to ignore the difficulty. For separated flows, however, the difficulty does not seem to be negligible. Perhaps the correct view of the difficulty in the separated flow case, is that the model cannot be generally valid for any reasonable choice of constitutive equations.

The point should be emphasized that Ishii's derivation is rigorous and precise. The entire problem is reduced to the determination of appropriate constitutive equations. The above argument suggests that, at least in some cases, it may not be possible to determine general constitutive equations.

Therefore, constitutive equations are formulated with specific flow regimes in mind, and results derived from the resulting conservation equations should be interpreted appropriately.

If the flow is well mixed, such as bubbly flows or droplet flows, then Ishii's time averaging procedure gives averaged quantities which are straightforward to interpret. Therefore, the task of determining appropriate constitutive equations is conceptually easier than in the case of separated flow.

For a well mixed flow, it can be assumed that enough interfacial points intersect the point $\underset{\sim}{x}$, during the averaging time interval from t to $t+\Delta t$, that the time average of the interfacial stress is equal to the average value of the stress over the surface of a typical "particle". (We shall use the term particle to denote dispersed phase entities in the well mixed flow.) When this assumption about the averaging can be made, formulating a constitutive relation for the interfacial force on the dispersed phase becomes a task of generalizing the force on a single particle to the appropriate two-phase flow situation.

Furthermore, when the flow is well mixed, the correlation of the velocity fluctuations, $-\overline{\rho_k \underset{\sim}{v}'_k \underset{\sim}{v}'_k}$, may be more straightforward to model than in the separated flow case. For instance, in the separated flow case, the velocity fluctuations may represent certain specific motions, such as wave motions caused by a Kelvin–Helmholtz instability. These specific fluctuations must be considered when constitutive equations are determined for the Reynolds stresses. When the flow is well mixed, the velocity fluctuations should indeed be random.

Consider the interfacial momentum transfer $\underset{\sim}{M}_k$. Ishii writes

$$\underset{\sim}{M}_k=\underset{\sim}{M}_k^{\Gamma}+\underset{\sim}{M}_k^{n}+\underset{\sim}{M}_k^{t}+\bar{\bar{p}}_{k_i}\nabla\alpha_k, \qquad [11]$$

where

$$\underset{\sim}{M}_k^{\Gamma}=\Gamma_k\bar{\underset{\sim}{v}}_{k_i}, \qquad [12]$$

$$\underline{M}_k^n = \frac{1}{\Delta t}\sum_j \frac{1}{v_{n_j}}[(\bar{\bar{p}}_{k_i} - p_k)\underline{n}_k + \tau_{n_k}], \quad [13]$$

$$\underline{M}_k^t = \frac{1}{t}\sum_j \frac{1}{v_{n_j}}\tau_{t_k}. \quad [14]$$

We note that $\underline{M}_k^\Gamma$ represents a momentum source to phase k due to change of phase. Since we are not considering phase change, we shall take $\underline{M}_k^\Gamma = 0$. Furthermore, the combination $\underline{M}_k^n + \underline{M}_k^t + \bar{\bar{p}}_{k_i}\nabla\alpha_k$ is the average interfacial force on phase k due to the stress forces on the interface. Ishii separates the average pressure force at the interface $\bar{\bar{p}}_{k_i}\nabla\alpha_k$ from the rest of this force, and combines $\underline{M}_k^n$ and $\underline{M}_k^t$ into a force $\underline{M}_k^d$, where

$$\underline{M}_k^d \overset{\Delta}{=} \underline{M}_k^t + \underline{M}_k^n. \quad [15]$$

Ishii treats $\underline{M}_k^d$ as a drag force, but it clearly contains other forces as well. Specifically, the drag force on a particle is due to the viscous forces at the surface of the particle, plus a contribution due to the net variation of the pressure from its interfacial average (form drag). Analysis of the virtual mass force indicates that acceleration effects also cause a variation of the pressure from its interfacial average. Hence [15] also contains virtual mass effects. Moreover, lift effects are also included in [15].

3. CONSTITUTIVE EQUATIONS

Constitutive equations describe the behavior of ideal materials by specifying how the material interacts with itself. In two-phase flows, we wish to describe the behavior of the two-phase mixture by specifying how each phase interacts with itself and with the other phase. Indeed, the important information about the microscopic details and profile effects which are lost during averaging must be replaced to some degree in the constitutive equations.

Two phase flows are further complicated by the appearance of flow regimes. If a general constitutive approach is to succeed, it must be assumed that enough information to specify flow regime must be contained in the *averaged* variables. It has not been established whether it is possible to describe flow regimes with multidimensional averaged variables.

Constitutive equations are needed for $\underline{M}_k^d$, $\underline{M}_m$, $\bar{\bar{p}}_k - \bar{\bar{p}}_{k_i}$, $\bar{\bar{p}}_1 - \bar{\bar{p}}_2$, $\bar{\bar{\underline{\tau}}}_k$ and $\underline{\tau}_k^T$. It is normally assumed that if the averaged fields are known, the values of the above variables can be determined. This assumption is little more than hope, but it is crucial to making predictions with averaged equations.

In this section, we shall first review the general principles used in formulating constitutive equations. We shall then apply these principles to the two-phase flow situation in order to obtain general forms for the variables to be determined. We shall then discuss the simplification of these general forms into useful constitutive equations.

The basic principles for formulating constitutive equations (Truesdell & Noll 1965) are:

(1) *Coordinate invariance*

This principle states that constitutive rules must be stated in a way which does not depend on the coordinate system. An example of a "constitutive equation" which *violates* this principle is

$$M_{1_x}^d = 0,$$

$$M_{1_y}^d = 0,$$

$$M_{1_z}^d = b^M(\bar{v}_{2_z} - \bar{v}_{1_z}), \quad [16]$$

for some given Cartesian coordinate system (x, y, z). The *correct* expression for such a flow would be $\underset{\sim}{M}_1{}^d = b^M(\underset{\sim}{\bar{v}}_2 - \underset{\sim}{\bar{v}}_1)$. It may well turn out that correctly formulated constitutive equations, when used in a specific flow situation, produce *solutions* of the form[16]. Care must be taken to avoid using any prior knowledge of the *solution* in formulating constitutive equations. The constitutive equations must be able to represent the most general situation, subject to the dependences proposed. One way to insure that the principle of coordinate invariance is satisfied is to work with the dyadic or invariant notation.

(2) *Equipresence*

This principle states that if one variable is known to depend on one specific field variable, then *all* other variables to be constituted must be allowed to depend on the same variable. For example, if we believe that $\underset{\sim}{M}_k{}^d$ depends on $\underset{\sim}{\bar{v}}_1 - \underset{\sim}{\bar{v}}_2$, then we must allow $\bar{\bar{\tau}}_k$, $\underset{\approx}{\tau}_k{}^T$, $\bar{\bar{p}}_k - \bar{\bar{p}}_{k_i}$ and $\bar{\bar{p}}_1 - \bar{\bar{p}}_2$ to also depend on $\underset{\sim}{\bar{v}}_1 - \underset{\sim}{\bar{v}}_2$. This principle prevents *a priori* prejudicing of the constitutive equations by selectively excluding certain dependencies. It is this principle which makes the general approach impractical in the final analysis, since it forces us to include dependencies for which there is no physical evidence. In our application of this principle to two-phase flows, we shall give the general forms for the constitutive equations satisfying this principle, but then retain only those terms for which there is some physical (or intuitive) reason that they should be included. This gives confidence in the constitutive equations so derived, since they contain known physical effects, but are still derived from the general forms.

(3) *Material frame indifference* (*objectivity*)

This principle states that variables, for which constitutive equations are needed, cannot depend on the coordinate frame in which the variables are expressed. (Here, "coordinate frame" denotes Euclidean three-space, plus time.) Coordinate frames are objects constructed by human beings to quantify motions. The materials which are undergoing the motions do not recognize these frames.

(4) *Homogeneity*

This principle states that constitutive equations which express the different behavior of different materials must do so by following those materials. For example, if a bar is made of steel for $-L < x < 0$, and made of aluminum for $0 < x < L$ at $t = 0$, then the appropriate stress–strain law for each material must be used in that material. Thus the modulus of elasticity would depend on position, but only in such a way that if the motion of each point of the bar were known, the modulus of elasticity appropriate to an arbitrary point in the bar at a given time could be determined by tracing back the history of that material to determine whether it were steel or aluminum. If the bar were *homogeneous* (all steel, for example) then the modulus of elasticity would be independent of position. As an example of a situation where such consideration may be required in two-phase flows, consider a tank of water, with air being injected in one place, and steam being injected in another. The air–water mixture, and the steam–water mixture have different characteristics, and different constitutive assumptions are needed for each. In this case, the region occupied by steam–water mixture, and the region occupied by the air–water mixture must be tracked.

(5) *Isotropy*

If neither of the two fluids making up the two-phase mixture has a preferred direction, the two-phase mixture is isotropic. We note that in a specific flow situation, a flow possessing a preferred direction may arise. Indeed, a shearing flow in a single phase viscous fluid shows preferred, or special, directions; but the material itself does not. We emphasize that it is essential to separate the process of determining constitutive equations from the solution of the equations of motion.

(6) *Just setting*

This principle states that the equations of motion, supplemented by the constitutive equations, and appropriate initial and boundary conditions, give solutions which are unique. It is extremely difficult to verify that this principle is satisfied.

(7) *Dimensional invariance*

This principle states that the constitutive equations must be dimensionally correct, and that arbitrary functional dependences can only occur through dimensionless variables.

In addition to the above general constitutive principles, there is a further principle which can be applied to two-phase flows.

(8) *Correct low concentration limits*

This principle states that, in the limit as $\alpha_k \rightarrow 0$, the equation of motion for phase k approaches the appropriate single particle equation, with the equations for the other phase approaching the correct equations for that single phase fluid. This principle suggests the appropriate variables to use in the constitutive equations. We shall also use it to establish physical reasons for retaining various terms when we proceed from the general constitutive equations to the equations valid for specific flow regimes.

It is not sufficient to postulate forms for the constitutive equations, and apply the principle of correct low concentration limits to determine the coefficients involved in these constitutive equations. As with any continuum mechanical theory, the coefficients in the constitutive equations must be evaluated by comparing with carefully done experiments done in simple geometries involving simple flow characteristics. On occasion, the required observations can be made by "thought experiments". Nevertheless, there can never be too many experiments which attempt to verify the constitutive equations, or to evaluate the coefficients which appear therein.

The constitutive principles are applied to two-phase flows, in order to determine functional forms for:

$$\underline{M}_k{}^d, \underline{M}_m, \bar{\bar{\underline{\underline{\tau}}}}_k, \underline{\underline{\tau}}_k{}^T, \bar{\bar{p}}_k - \bar{\bar{p}}_{k_i}, \bar{\bar{p}}_1 - \bar{\bar{p}}_2 \qquad [17]$$

in terms of

$$\alpha_k, \partial\alpha_k/\partial t, \nabla\alpha_k, \bar{\underline{v}}_k, \nabla\bar{\underline{v}}_k, \partial\bar{\underline{v}}_k/\partial t, \ldots, \qquad [18]$$

where ... represent microscopic quantities, such as the exact viscosities and densities of the two fluids involved, and other geometric parameters, such as the average bubble or droplet radius, or the interfacial area density. It should be noted that the general theory does not need explicit information on flow regime; however, the specific variables in [18] may indeed specify flow regime.

The specific quantities displayed in [18] include all the vectorial and tensorial quantities needed to formulate a theory which is first order in space (i.e. involves first derivatives in the spatial coordinates) and is first order in time in the velocity variables. Many of the forces known or suspected to be important in two-phase flows are included in this theory.

The velocity variables in [18] cannot appear arbitrarily in the constitutive equations, since the constitutive equations must be objective.

The concept of objectivity is now examined. Consider a coordinate change from system $\underline{x}$ to system $\underline{x}'$, specified by

$$\underline{x}' = \underline{\underline{Q}}(t) \cdot \underline{x} + \underline{b}(t), \qquad [19]$$

where $\underset{=}{Q}(t)$ is an orthonormal tensor, and $\underline{b}(t)$ is a vector. A scalar quantity is objective if $\phi' = \phi$; that is, if the value of ϕ is the same in the primed system as in the unprimed system. A vector is objective if it transforms coordinates correctly; that is, if $\underline{v}' = \underset{=}{Q} \cdot \underline{v}$. A tensor is objective if $\underset{=}{T}' = \underset{=}{Q} \cdot \underset{=}{T} \cdot \underset{=}{Q}^T$.

The scalar functions α_k and r_d (where r_d is the radius of the dispersed phase, where applicable) are objective. The partial derivatives with respect to t are not, however. To see this, consider a scalar function $\phi(\underline{x}, t)$. In terms of the primed coordinates, we have

$$\phi'(\underline{x}', t) = \phi(\underline{x}, t). \qquad [20]$$

Taking the partial derivative of ϕ, keeping $\underline{x}$ constant gives

$$\begin{aligned} \frac{\partial \phi}{\partial t}(\underline{x}, t)\bigg|_{\underline{x}} &= \frac{\partial \phi'}{\partial t}(\underline{x}', t)\bigg|_{\underline{x}} = \frac{\partial \phi'}{\partial t}(\underset{=}{Q}(t) \cdot \underline{x} + \underline{b}(t), t)\bigg|_{\underline{x}} \\ &= \frac{\partial \phi'}{\partial t}\bigg|_{\underline{x}'} + \frac{\partial x_i'}{\partial t}\bigg|_{\underline{x}} \frac{\partial \phi}{\partial x_i'}\bigg|_t \\ &= \frac{\partial \phi}{\partial t} + (\dot{\underset{=}{Q}} \cdot \underline{x} + \dot{\underline{b}}) \cdot \nabla' \phi'. \end{aligned} \qquad [21]$$

Therefore $\partial \phi / \partial t$ is not objective.

It is relatively straightforward to show that $D_j\phi/Dt \overset{\Delta}{=} \partial \phi/\partial t + \underline{v}_j \cdot \nabla \phi$ is objective, using arguments similar to that in [21], and those below, for the velocity. We shall take a different approach here, which involves changing ϕ to material coordinates. For this, we have a motion given by $\underline{x} = \underline{x}(\underline{A}, t)$, where $\underline{x}(\underline{A}, 0) = \underline{A}$. We define $\Phi(\underline{A}, t) = \phi(\underline{x}(\underline{A}, t), t)$. Then

$$\phi'(\underline{x}', t) = \Phi'(\underline{A}', t) = \Phi(\underline{A}, t) = \phi(\underline{x}, t), \qquad [22]$$

where $\underline{A}'$ is the initial location (at $t = 0$) of the particle now at $\underline{x}'$. We note that

$$\underline{A}' = \underset{=}{Q}(0) \cdot \underline{A} + \underline{b}(0). \qquad [23]$$

$$\frac{\partial \Phi}{\partial t}(\underline{A}, t) = \frac{\partial \Phi'}{\partial t}(\underline{A}', t)\bigg|_{\underline{A}} = \frac{\partial \Phi'}{\partial t}(\underline{A}', t)\bigg|_{\underline{A}'}, \qquad [24]$$

since $\partial \underline{A}'/\partial t = 0$. Thus, the material derivative of a scalar is objective, so that

$$D_j\alpha_k/Dt, \; D_jr_d/Dt \qquad [25]$$

are objective.

Next, objective combinations of the vectors are considered. Differentiate [9] with respect to t, following a material particle of phase k:

$$\bar{\underline{v}}_k' \overset{\Delta}{=} \frac{D_k\underline{x}'}{Dt} = \dot{\underset{=}{Q}}(t) \cdot \underline{x} + \underset{=}{Q} \cdot \bar{\underline{v}}_k + \dot{\underline{b}}. \qquad [26]$$

Thus, in general, $\bar{\underline{v}}_k' \neq \underset{=}{Q} \cdot \bar{\underline{v}}_k$. Hence $\bar{\underline{v}}_k$ is not objective. This is also obvious on physical grounds, since the velocity observed for some object depends on the motion of the reference frame.

It is easily seen that relative velocity is objective. To see this, consider[26] for $k = 1$, and $k = 2$, and subtract one from the other, yielding

$$\underline{\bar{v}}_1' - \underline{\bar{v}}_2' = (\dot{\underline{\underline{Q}}} \cdot \underline{x} + \underline{\underline{Q}} \cdot \underline{\bar{v}}_1 + \dot{\underline{b}}) - (\dot{\underline{\underline{Q}}} \cdot \underline{x} + \underline{\underline{Q}} \cdot \underline{\bar{v}}_2 + \dot{\underline{b}}) = \underline{\underline{Q}} \cdot (\underline{\bar{v}}_1 - \underline{\bar{v}}_2). \qquad [27]$$

Obviously, $\underline{v}_r \overset{\Delta}{=} \underline{\bar{v}}_1 - \underline{\bar{v}}_2$ is objective.

A combination of accelerations which is objective is sought. Consider

$$\begin{aligned} \frac{D_1\underline{\bar{v}}_2'}{Dt} - \frac{D_2\underline{\bar{v}}_1'}{Dt} &= \left(\ddot{\underline{\underline{Q}}} \cdot \underline{x} + \dot{\underline{\underline{Q}}} \cdot \underline{\bar{v}}_1 + \dot{\underline{\underline{Q}}} \cdot \underline{\bar{v}}_2 + \underline{\underline{Q}} \cdot \frac{D_1\underline{\bar{v}}_2}{Dt} + \ddot{\underline{b}}\right) \\ &\quad - \left(\ddot{\underline{\underline{Q}}} \cdot \underline{x} + \dot{\underline{\underline{Q}}} \cdot \underline{\bar{v}}_2 + \dot{\underline{\underline{Q}}} \cdot \underline{\bar{v}}_1 + \underline{\underline{Q}} \cdot \frac{D_2\underline{\bar{v}}_1}{Dt} + \ddot{\underline{b}}\right) \\ &= \underline{\underline{Q}} \cdot \left(\frac{D_1\underline{\bar{v}}_2}{Dt} - \frac{D_2\underline{\bar{v}}_1}{Dt}\right). \end{aligned} \qquad [28]$$

Thus

$$\underline{a}_{12} \overset{\Delta}{=} \frac{D_1\underline{\bar{v}}_2}{Dt} - \frac{D_2\underline{\bar{v}}_1}{Dt} = \left(\frac{\partial \underline{\bar{v}}_2}{\partial t} + \underline{\bar{v}}_1 \cdot \nabla \underline{\bar{v}}_2\right) - \left(\frac{\partial \underline{\bar{v}}_1}{\partial t} + \underline{\bar{v}}_2 \cdot \nabla \underline{\bar{v}}_1\right) \qquad [29]$$

is objective.

Consider objective combinations of the velocity gradients. By taking the gradient (with respect to $\underline{x}'$) of[26], and using the result that

$$\nabla' \underline{x} = \underline{\underline{Q}}^T, \qquad [30]$$

we have

$$\nabla' \underline{\bar{v}}_k' = \underline{\underline{Q}} \cdot \nabla \underline{\bar{v}}_k \cdot \underline{\underline{Q}}^T + \underline{\underline{Q}} \cdot \dot{\underline{\underline{Q}}}^T. \qquad [31]$$

Taking the transpose of[31],

$$\underline{\bar{v}}_k' \nabla' \overset{\Delta}{=} (\nabla' \underline{\bar{v}}_k')^T = \underline{\underline{Q}} \cdot (\underline{\bar{v}}_k \bar{\nabla}) \cdot \underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}} \cdot \underline{\underline{Q}}^T. \qquad [32]$$

But $\underline{\underline{Q}} \cdot \underline{\underline{Q}}^T = \underline{\underline{I}}$, so $\dot{\underline{\underline{Q}}} \cdot \underline{\underline{Q}}^T + \underline{\underline{Q}} \cdot \dot{\underline{\underline{Q}}}^T = 0$. Thus it is evident that

$$D_{k_b} \overset{\Delta}{=} \frac{1}{2}[\nabla \bar{v}_k + (\nabla \underline{\bar{v}}_k)^T] \qquad [33]$$

is objective. Furthermore,

$$\underline{\underline{\mathscr{D}}}_{12} = \frac{1}{2}[\nabla \underline{\bar{v}}_1 + (\nabla \underline{\bar{v}}_2)^T] \qquad [34]$$

$$\underline{\underline{\mathscr{D}}}_{21} = \frac{1}{2}[\nabla \underline{\bar{v}}_2 + (\nabla \underline{\bar{v}}_1)^T] \qquad [35]$$

are also objective. Note that $\underline{\underline{\mathscr{D}}}_{12} + \underline{\underline{\mathscr{D}}}_{21} = \underline{\underline{D}}_{1b} + \underline{\underline{D}}_{2b}$. Thus only three of the tensors $\underline{\underline{D}}_{1b}$, $\underline{\underline{D}}_{2b}$, $\underline{\underline{\mathscr{D}}}_{12}$ and $\underline{\underline{\mathscr{D}}}_{21}$ are independent. Furthermore, it is more convenient to work with the objective tensor

$$\underline{\underline{D}}_{12} \overset{\Delta}{=} \nabla(\underline{\bar{v}}_1 - \underline{\bar{v}}_2). \qquad [36]$$

We note that

$$\underline{\underline{D}}_{12} = 2(\underline{\underline{D}}_{1b} - \underline{\underline{\mathscr{D}}}_{21}) \qquad [37]$$

Thus we may replace[18] by

$$\alpha_1, D_k\alpha_1/Dt, \nabla\alpha_1, \underline{\bar{v}}_1 - \underline{\bar{v}}_2, \underline{a}_{12}, \underline{\underline{D}}_{1b}, \underline{\underline{D}}_{2b}, \underline{\underline{D}}_{12}, \ldots. \qquad [38]$$

These variables, and combinations thereof, are objective.

Constitutive equations determined from[38] are objective. More information can be determined from consideration of the vectorial or tensorial nature of the variables to be constituted.

If f is a scalar function, then f can depend only on scalar invariants formed from[38]. The complete list of scalar invariants which can be formed from[38] is given in the appendix. Basically, any scalar which can be formed from the vectors or tensors in[38] by invariant operations is a scalar invariant. Thus the length of a vector is a scalar invariant. Also, the dot or scalar product of two vectors is a scalar invariant. For a tensor $\underline{\underline{T}}$, there are three scalar invariants, $I_{\underline{\underline{T}}} = \text{tr}\ \underline{\underline{T}}$, $II_{\underline{\underline{T}}} = \underline{\underline{T}} : \underline{\underline{T}}$, and $III_{\underline{\underline{T}}} = \det \underline{\underline{T}}$. Thus

$$f = f(S_1, S_2, \ldots), \qquad [39]$$

where S_i are the scalar invariants given in the Appendix.

If $\underline{F}$ is a vector function, then $\underline{F}$ must be linear in all the vectors which can be formed in an invariant way from[38]. Thus

$$\underline{F} = \sum_i a_i \underline{v}_i, \qquad [40]$$

where $\underline{V}_i$ are the possible objective vectors given in the Appendix. Each scalar coefficients a_i can be a function of the scalar invariants S_j.

If $\underline{\underline{\mathscr{F}}}$ is a second order tensor, then

$$\underline{\underline{\mathscr{F}}} = \sum_j B_j \hat{\underline{\underline{T}}}_j \qquad [41]$$

where $\hat{\underline{\underline{T}}}_j$ are the objective tensors given in the Appendix. Again, the scalar coefficients B_j are each a function of the scalar invariants S_i.

4. SPECIFIC FORMS FOR CONSTITUTIVE EQUATIONS

As shown in the Appendix, if we were to use this general approach to give forms for the constitutive variables in[17], we would be required to determine 2771 scalar functions of the 679 scalar variables! Therefore, it seems impractical to attempt to carry out a completely general approach any further. We therefore propose that the principle of equipresence must be compromised. We shall, however, retain the rest of the general constitutive formalism. Thus, we seek to formulate constitutive equations which satisfy all the general constitutive principles except equipresence. Our approach for the remainder of this paper gives constitutive relations specifying the behavior of specific two-phase mixtures which obey all the constitutive principles except equipresence.

Consider a phenomenological approach, to determine useful constitutive equations within the framework of rational mechanics. In this approach, combinations which have shown to be important in experiments or in specific solutions to the equations for the flow around particles, are included. The other possible terms in the constitutive equations are then ignored. While this approach can produce good results when applied to certain flow situations, it can break down in others if important terms have been neglected.

4.1 *Stresses*

Consider the stress tensors $\bar{\underline{\underline{\tau}}}_k$ and $\bar{\underline{\underline{\tau}}}_k{}^T$. For the laminar stresses, Ishii (1975) applies the time average to the exact stress. Therefore, we have

$$\begin{aligned}\bar{\underline{\underline{\tau}}}_k &= \mu_k(\overline{\overline{\nabla \underline{v}_k + \underline{v}_k \nabla}}) \\ &= \mu_k(\nabla \bar{\underline{v}}_k + \bar{\underline{v}}_k \nabla) + \frac{1}{\alpha_k}\frac{1}{\Delta t}\sum_j \frac{1}{v_{n_i}}(\underline{n}_k \underline{v}'_k + \underline{v}'_k \underline{n}_k) \\ &= 2\mu_k(\underline{\underline{D}}_{k_b} + \underline{\underline{D}}_{k_i}),\end{aligned} \qquad [42]$$

where

$$\underline{\underline{D}}_{k_i} \overset{\Delta}{=} \frac{1}{2\alpha_k}\frac{1}{\Delta t}\sum_j \frac{1}{v_{n_i}}(\underline{n}_k \underline{v}'_k + \underline{v}'_k \underline{n}_k).$$

We must now specify a constitutive relationship for $\underline{\underline{D}}_{k_i}$.

Equation[41] shows that $\underline{\underline{D}}_{k_i}$ can depend on many tensorial quantities. Ishii looks at the specific situation where one phase (in the form of either droplets or bubbles) is dispersed. For $k = d$, we have

$$D_{d_i} \cong 0. \qquad [43]$$

Also, using $\bar{\underline{v}}'_c \cong \bar{\underline{v}}_c - \bar{\underline{v}}_d$, we have

$$\underline{\underline{D}}_{c_i} = -\frac{1}{2\alpha_c}[\nabla \alpha_c(\bar{\underline{v}}_d - \bar{\underline{v}}_c) + (\bar{\underline{v}}_d - \bar{\underline{v}}_c)\nabla \alpha_c]. \qquad [44]$$

While it seems unlikely that[43] and [44] would hold in general, the form of[44] suggests that

$$\underline{\underline{D}}_{k_i} = -\frac{b(\alpha_k)}{2\alpha_k}[\nabla \alpha_k(\bar{\underline{v}}_j - \bar{\underline{v}}_k) + (\bar{\underline{v}}_j - \bar{\underline{v}}_k)\nabla \alpha_k] \qquad [45]$$

where $j = 1$ if $k = 2$, and $j = 2$ if $k = 1$. Ishii calls the term $b_k(\alpha)$ the mobility of phase k.

Consider the Reynolds stresses $\underline{\underline{\tau}}_k{}^T$. There are practically no experiments to determine the form of these tensors. The general approach,[41] gives 674 coefficients, which need to be evaluated to specify $\underline{\underline{\tau}}_k{}^T$. Of the few observations which have been made, none have attempted to determine values of the coefficients in the form given by[41]. Ishii postulates that

$$\underline{\underline{\tau}}_k{}^T = a_{k_0}\underline{\underline{I}} + a_{k_1}\underline{\underline{D}}_k + a_{k_2}\underline{\underline{D}}_k \cdot \underline{\underline{D}}_k, \qquad [46]$$

where $\underline{\underline{D}}_k = \underline{\underline{D}}_{k_b} + \underline{\underline{D}}_{k_i}$. This is a generalization of Prandtl's mixing length argument. Ishii further argues that

$$\underline{\underline{\tau}}_k^T = 2\mu_k^T \underline{\underline{D}}_k, \qquad [47]$$

where μ_k^T is the eddy viscosity, and can depend on the scalar invariants.

Both the forms [46] and [47] have deficiencies in some important flow situations. We note that if the flow is undirectional shear flow, with $\bar{\underline{v}}_k = \bar{v}_k(y)\underline{i}$, and $\alpha_k = \alpha_k(y)$, then both $\underline{\underline{D}}_{k_b}$ and $\underline{\underline{D}}_{k_i}$ are proportional to the tensor $\underline{i}\underline{j} + \underline{j}\underline{i}$. In this case, the diagonal terms, as given by [47], are zero. The diagonal terms of $\underline{\underline{\tau}}_k^T$, given by

$$\tau_{k_{ii}} = -\overline{\overline{\rho_k v'^2_{k_i}}} \qquad \text{(no sum on } i\text{)} \qquad [48]$$

can be zero only if the flow is laminar, since, if $v'_{k_i} = 0$, then $v_{k_i} = \bar{v}_{k_i}$, and there are no velocity fluctuations. Thus the simple model [47], is obviously incorrect. We further note that the diagonal terms of $\underline{\underline{\tau}}_k^T$ are important in determining multidimensional effects. Drew *et al.* (1978) have shown the importance of the liquid phase turbulent kinetic energy in determining the radial distribution of the gas phase in the steady, fully developed concurrent flow of a bubbly mixture of air and water in a circular pipe.

Consider [46] in the situation of gas bubbles rising uniformly through quiescent liquid. The passage of the bubbles generates liquid phase velocity fluctuations which must appear in the term $\underline{\underline{\tau}}_l^T$. Ishii's simple model, [47], gives no Reynolds stresses in this flow. Ishii's more general model, [46], gives $\underline{\underline{\tau}}_l^T = a_{l_0}\underline{\underline{I}}$, so that the Reynolds stress is isotropic. Since the flow is expected to be different in the direction of rise, from the direction perpendicular to the rise direction, Ishii's model seems inadequate.

The invariant quantity which picks out the direction of rise is $\bar{\underline{v}}_1 - \bar{\underline{v}}_2$. Also, $\nabla\alpha_1$ is an invariant quantity. Thus a better constitutive model should be

$$\underline{\underline{\tau}}_k^T = a_{k_0}\underline{\underline{I}} + a_{k_1}\underline{\underline{D}}_k + b_1(\bar{\underline{v}}_1 - \bar{\underline{v}}_2)(\bar{\underline{v}}_1 - \bar{\underline{v}}_2) + b_2\nabla\alpha_1\nabla\alpha_1 \qquad [49]$$

where a_{k_0}, a_{k_1}, b_1 and b_2 may be functions of α_1, $|\underline{v}_r| I_{D_k}$, II_{D_k}, $III_{D_k}, \ldots$. This retains the term which is believed to adequately model the shearing effects, in the form $a_{k_1}\underline{\underline{D}}_k$. Moreover, it corrects this by adding terms which model the diagonal terms of $\underline{\underline{\tau}}_k^T$, and allow a difference in the diagonal element in the directions of relative velocity and the volumetric vapor (i.e. void) fraction gradient.

4.2 *Interfacial force*

Consider the interfacial force density $\underline{M}_k$. Equations [11]–[15] suggest that a constitutive model is needed for $\underline{M}_k^d$, $k = 1$ and 2, or equivalently† for $\underline{M}_1^d$ and $\underline{M}_m$. (Recall we are assuming that $\underline{M}_k^\Gamma = 0$.)

Constitutive equations are now formulated for the volumetric interfacial force on the dispersed phase, $\underline{M}_d^d$, which are appropriate when one phase is dispersed in the other (such as bubbles in liquid, or droplets in gas).

Since $\underline{M}_d^d$ is a vector, the general constitutive equation for it is given by [40]. However, not all of the terms in [40] correspond to forces which have been observed or calculated.

We now postulate a constitutive relation for the volumetric interfacial force on the dispersed phase which avoids this difficulty. To do this, we shall treat the force on the dispersed phase, *plus the interface*. Thus, the force for which we now derive a constitutive equation is

†See [6].

$-\underline{M}_c^d = \underline{M}_d^d + \underline{M}_m$. In the absence of surface tension, $\underline{M}_m = 0$, and the force becomes $\underline{M}_d^d$. Thus we postulate

$$-\underline{M}_c^d = A_1(\bar{\underline{v}}_c - \bar{\underline{v}}_d) + A_2\underline{a}_{cd} + A_3(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot\underline{\underline{D}}_{c_b} + A_4(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot\underline{\underline{D}}_{db}$$
$$+ A_5(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot\underline{\underline{D}}_{cd} + A_6\underline{\underline{D}}_{cd}\cdot(\bar{\underline{v}}_c - \bar{\underline{v}}_d) \quad [50]$$

where $A_{1\text{-}6}$ are scalar functions of the invariants.

The first term in [50] represents the classical drag forces. It is customary to write

$$A_1 = \frac{3}{8}\alpha_d\rho_c\frac{C_D}{r_d}|\bar{\underline{v}}_c - \bar{\underline{v}}_d|, \quad [51]$$

where r_d is the effective radius of the dispersed phase, and the drag coefficient, C_D, is a function of the invariants. It is usually assumed that $C_D = C_D(\alpha_d, \mathrm{Re}_d)$, where $\mathrm{Re}_d \overset{\Delta}{=} 2\rho_c|\bar{\underline{v}}_c - \bar{\underline{v}}_d|r_d/\mu_c$ is the particle Reynolds number (e.g. Zuber & Ishii 1978). It is not clear, however whether such models are valid for multidimensional application. The models are usually verified by comparing the relative velocity observed in various flows to the relative velocity predicted using a balance between buoyancy and drag. Resulting correlations have been checked against many sets of data, however; almost all of these data have been global, and therefore may be more appropriate for cross-sectionally averaged drag models than local models. Lahey *et al.* (1979) have shown that the existing correlations do not capture all the detailed local data trends, when compared to the bubbly flow data taken by Serizawa (1974) at different radial positions in a circular pipe. Figures 1 and 2 indicate the difficulty. We note that $\underline{M}_v/\alpha v_r^2 = 3C_D/8r_b\rho_l$ was determined from Serizawa's data by calculating the buoyancy, and turbulent and laminar stresses at each radial position. Figures 1 and 2 indicate that the measured dependence of C_D on α_d brackets two well-known correlations (Wallis 1976, Zuber & Ishii 1978). A definite trend in the data is not well represented in these correlations, however. Specifically, the data show that C_D is lower than the value given by the correlations near the centerline but higher than the value given by the correlations near the wall. It is not known at this time whether this is due to an inadequate assumption about the variables of importance in C_D, or whether an important term is missing from [50].

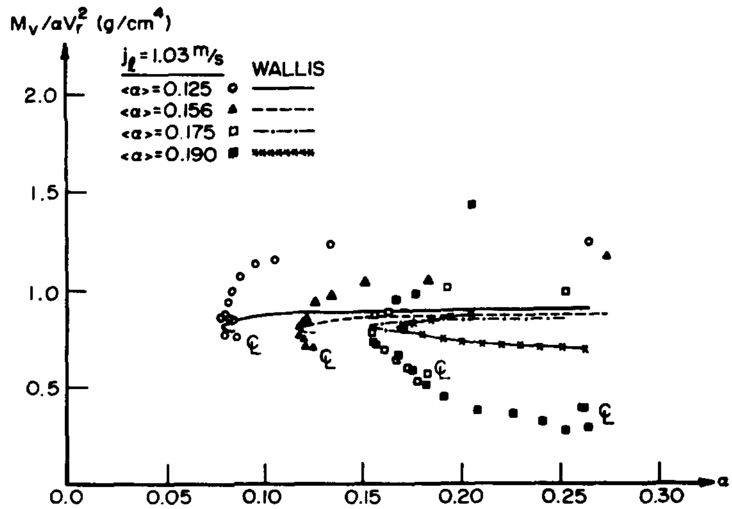

Figure 1. Comparison of Wallis' (1976) "dirty water" drag models with Serizawa's (1974) data.

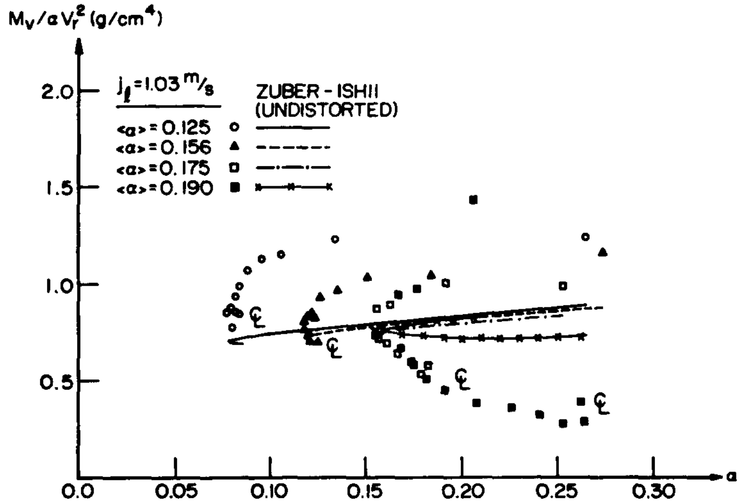

Figure 2. Comparison of Zuber & Ishii's (1978) undistorted bubble drag model with Serizawa's (1974) data.

The collective terms

$$A_2\underline{a}_{cd} + A_5(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot \underline{\underline{D}}_{cd} = A_2\left[\left(\frac{\partial \bar{\underline{v}}_c}{\partial t} + \bar{\underline{v}}_d\cdot\nabla\bar{\underline{v}}_c\right) - \left(\frac{\partial \bar{\underline{v}}_d}{\partial t} + \bar{\underline{v}}_c\cdot\nabla\bar{\underline{v}}_d\right)\right]$$
$$+ A_5(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot\nabla(\bar{\underline{v}}_c - \bar{\underline{v}}_d) \qquad [52]$$

are referred to by Drew *et al.* (1979) as the volumetric virtual mass force. This particular term is the most general *acceleration* force which can be included in $\underline{M}_k$. There are convective accelerations in the remaining terms in[50], but these are necessarily coupled with non-convective terms. For example, consider the term

$$A_3(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot \underline{\underline{D}}_{c_b} = \frac{1}{2}A_3(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot\nabla\bar{\underline{v}}_c + \frac{1}{2}A_3(\nabla\bar{\underline{v}}_c)\cdot(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\,. \qquad [53]$$

In[53], the term $(1/2)A_3(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot\nabla\bar{\underline{v}}_c$ can be viewed as a convective acceleration, but $(1/2)A_3(\nabla\bar{\underline{v}}_c)\cdot(\bar{\underline{v}}_c - \bar{\underline{v}}_d)$ can not.

For a single spherical particle in a nearly inviscid quiescent fluid, a force on the sphere is generated when the sphere accelerates. This force is equal to one half the mass of the fluid displaced by the particle, times the acceleration of the center of mass of the sphere. This argument suggests that

$$A_2 = \frac{1}{2}\alpha_d\rho_c \qquad [54]$$

for small α_d. Drew *et al.* (1979) write

$$A_2 = \alpha_d\rho_c C_{VM}(\alpha_d) \qquad [55]$$

for the more general case. Houghton's (1978) data give $C_{VM} \sim 0.1$–0.5 for various particles in accelerating flows. Drew *et al.* (1979) also write

$$A_5 = \alpha_d\rho_c C_{VM}(\alpha_d)\cdot(1 - \lambda(\alpha_d))\,, \qquad [56]$$

where $\lambda(\alpha_d)$ must be found by experimental observations. They assert that, as $\alpha_d \to 0$, $\lambda(\alpha_d) \to 2$, in order that the virtual mass acceleration reduce to the acceleration of a single particle in an infinite fluid. No appropriate observations have been found to compare with the form given by[50]. It is anticipated that observations on two component, two-phase critical flow may give information on $\lambda(\alpha_d)$, since large spatial accelerations are found in choked flow.

The term $(1/2)A_3(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot(\nabla\underline{v}_c)^T$ represents a force on the dispersed phase due to the interaction of the relative velocity and the shear of the continuous phase. Such a force can arise in slow viscous flow around a single sphere (Saffman 1965, 1968, Harper & Chang 1968, Drew 1978), where it is called a lift force. To see that $(1/2)A_3(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot(\nabla\underline{v}_c)^T$ is a lift force, consider the special flow situation when $\bar{\underline{v}}_k = \bar{v}_k(y)\underline{i}$. Then

$$(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot(\nabla\bar{\underline{v}}_c)^T = (\bar{v}_c - \bar{v}_d)\frac{\mathrm{d}\bar{v}_c}{\mathrm{d}y}\underline{j} \qquad [57]$$

is perpendicular to the direction of the relative velocity, and therefore is "lifting". It should be noted that the calculations mentioned above is for slow viscous flow. There is no obvious indication that the lift force is of practical importance in gas–liquid flows, where the flow is almost never slow viscous flow. We include this force because it is a known force in some flows. On the other hand, the terms $A_4(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot\underline{\underline{D}}_{d_b}$ and $A_6\underline{\underline{D}}_{cd}\cdot(\bar{\underline{v}}_c - \bar{\underline{v}}_d)$ have no analogs in single particle calculations, and will be neglected. Thus,[50] becomes,

$$\begin{aligned}\underline{M}_d^{\,d} = \frac{3}{8}\alpha_d\frac{C_D}{r_d}|\bar{\underline{v}}_c - \bar{\underline{v}}_d|(\bar{\underline{v}}_c - \bar{\underline{v}}_d) + \alpha_d\rho_c C_{VM}(\alpha_d)\Bigg[\left(\frac{\partial\bar{\underline{v}}_c}{\partial t} + \bar{\underline{v}}_d\cdot\nabla\bar{\underline{v}}_c\right) - \left(\frac{\partial\bar{\underline{v}}_d}{\partial t} + \bar{\underline{v}}_c\cdot\nabla\bar{\underline{v}}_d\right)\\ + (1-\lambda(\alpha_d))(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot\nabla(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\Bigg] + A_3(\bar{\underline{v}}_c - \bar{\underline{v}}_d)\cdot\underline{\underline{D}}_{cb}.\end{aligned} \qquad [58]$$

This equation is a constitutive equation for $\underline{M}_d^{\,d}$ which contains known drag, lift and virtual mass forces.

It is important to note which terms which are not included in[58]. First, no terms proportional to $\nabla\alpha_1$ have been included. There is no evidence that such a term is needed. Single- and multiple-sphere calculations do not indicate the need for such a term.

There are other known forces which are not included in[50], e.g. the Basset force.

The Basset force, for a single sphere in a still fluid, is (Soo 1967):

$$-\frac{3}{2}r_d^2\sqrt{(\pi\rho_c\mu_c)}\int_{-\infty}^{t}\frac{\mathrm{d}\underline{U}/\mathrm{d}t'}{\sqrt{(t-t')}}\mathrm{d}t' \qquad [59]$$

where $\underline{U}(t)$ is the velocity of the centre of the sphere. In general, such a force depends on the history of the particle and fluid motion. While it is within the scope of technical competence to include such terms, we choose not to do so in this paper. Indeed, to include history or memory effects systematically would be difficult, since so many variables are believed to be important in the constitutive equations.

Another known force which is not included in[44] is the Faxén force. The Faxén force on a single sphere in a viscous fluid is given by (Happel & Brenner 1965):

$$\pi r_d^3\mu_c\frac{\mathrm{d}^2\underline{V}}{\mathrm{d}y^2}, \qquad [60]$$

where $\underline{V}(y)$ is the fluid velocity profile far from the particle, which is assumed to be in linear shear, r_d is the sphere radius, and μ_c is the fluid viscosity. The Faxén force depends on the second spatial derivative of the fluid velocity far from the sphere. To include this force in a general development would require that terms proportional to the third order tensors $\nabla\nabla \underline{v}_c$ and $\nabla\nabla \underline{v}_d$ be included in the list of constitutive variables.

4.3 *Pressure differences*

Consider the scalar functions for which we must formulate constitutive relations. Within the framework of the general theory, $\bar{\bar{p}}_k - \bar{\bar{p}}_{k_i}$ can be a function of the 679 scalar invariants given in the Appendix.

It is customary to assume that $\bar{\bar{p}}_k - \bar{\bar{p}}_{k_i} = 0$. This is not generally true; for example, in situations where the bubbles can expand, it is well known that the Rayleigh equation determines the relationship between the interfacial pressure and the pressure far from the bubble. Thus the most general model for $\bar{\bar{p}}_k - \bar{\bar{p}}_{k_i}$ should contain the Rayleigh equation.

First, we note that when no motion is present, we must have $\bar{\bar{p}}_k = \bar{\bar{p}}_{k_i}$. However, if we assume that $\bar{\bar{p}}_k - \bar{\bar{p}}_{k_i} = f_k^\circ(\alpha_1, |\nabla\alpha_1|, r_d)$, then, necessarily, we must have $f_k^\circ \equiv 0$. Thus, a more general assumption is needed.

For many transients of practical significance we expect $\bar{\bar{p}}_d \cong \bar{\bar{p}}_{d_i}$ with the dispersed (vapor) phase. This is due to the fact that the dispersed phase occupies relatively small regions within which no large pressure gradients can be supported.

Thus, we shall assume that the expansion/contraction of bubbles can give a contribution to $\underline{p}_k - \underline{p}_{k_i}$ only in the continuous (liquid) phase.

We further note, from the continuous phase momentum equation[3], that the net volumetric force on the liquid phase due to the term $\bar{\bar{p}}_c - \bar{\bar{p}}_{c_i}$, is of the form $(\bar{\bar{p}}_c - \bar{\bar{p}}_{c_i})\nabla\alpha_c$. Thus, this force can have an effect only when $\nabla\alpha_c \neq 0$. This complicates our task, since, in order to see the Rayleigh effect, we must have a flow situation with expanding or contracting bubbles, and a non-zero gradient of the volumetric concentration of phase k.

The classical Rayleigh equation for the growth of a single bubble in an infinite liquid is

$$p_{l_\infty} - p_{l_i} = -\left\{\rho_l\left[r_b\frac{\mathrm{d}^2 r_b}{\mathrm{d}t^2} + \frac{3}{2}\left(\frac{\mathrm{d}r_b}{\mathrm{d}t}\right)^2\right] + \frac{\mu_l}{r_b}\frac{\mathrm{d}r_b}{\mathrm{d}t}\right\} \qquad [61]$$

where p_{l_∞} is the liquid pressure far from the bubble, and r_b is the bubble radius. For the more general case, we expect that $\bar{\bar{p}}_c - \bar{\bar{p}}_{c_i}$ should depend on $r_d'(\bar{x}_i, t)$, the radius of the dispersed phase associated with location $\underline{x}$, at time t. Also, the appropriate derivative to use should be D_d/Dt, the material derivative following the dispersed phase. Thus we assume,

$$\bar{\bar{p}}_c - \bar{\bar{p}}_{c_i} = f(r_d, D_d r_d/Dt, D_d^2 r_d/Dt^2, \ldots), \qquad [62]$$

where ... represents quantities which include the effective density and viscosity.

Since the interfacial pressure difference $\bar{\bar{p}}_{2_i} - \bar{\bar{p}}_{1_i}$ is a scalar, it can be a function of the 679 variables in the Appendix. We expect $\bar{\bar{p}}_{2_i} - \bar{\bar{p}}_{1_i}$ to be non-zero in situations where surface tension is important. The exact instantaneous expression which we wish to generalize for our constitutive equations is

$$p_{2_i} - p_{1_i} = \kappa\sigma, \qquad [64]$$

where κ is the mean curvature of the interface. Thus, we shall assume that $\bar{\bar{p}}_{2_i} - \bar{\bar{p}}_{1_i}$ is a function of the geometry of the interface, and the surface tension σ, but shall assume that $\bar{\bar{p}}_{2_i} - \bar{\bar{p}}_{1_i}$ does not depend on the time derivative terms $D_j\alpha_1/Dt$ or $D_j r_d/Dt$.

Thus we assume that

$$\bar{\bar{p}}_{2_i} - \bar{\bar{p}}_{1_i} = f_i(\alpha_1, |\nabla\alpha_1|, r_d, \bar{\bar{\sigma}}, \ldots), \qquad [65]$$

where $\bar{\bar{\sigma}}$ is the coefficient of surface tension. We note that in the absence of surface tension, we must have

$$\bar{\bar{p}}_{2_i} = \bar{\bar{p}}_{1_i}. \qquad [66]$$

If surface tension is not neglected, then the jump condition for momentum becomes $\underline{M}_1 + \underline{M}_2 = \underline{M}_m$, where $\underline{M}_m$ is the mixture momentum source, given by Ishii is

$$\underline{M}_m = 2\bar{\bar{H}}_{21}\bar{\bar{\sigma}}\nabla\alpha_2 + \underline{M}_m{}^H, \qquad [67]$$

where $\bar{\bar{\sigma}}$ is the coefficient of surface tension, which is assumed to be a constant, $\bar{\bar{H}}_{21}$ is the average curvature of the interface. The term $\underline{M}_m{}^H$ represents forces due to the changing curvature. We shall assume that $\underline{M}_m{}^H = 0$.

The jump condition for momentum becomes

$$\underline{M}_1{}^d + \bar{\bar{p}}_{1_i}\nabla\alpha_1 + \underline{M}_2{}^d + \bar{\bar{p}}_{2_i}\nabla\alpha_2 = 2\bar{\bar{H}}_{21}\bar{\bar{\sigma}}\nabla\alpha_2. \qquad [68]$$

When no motion is present, $\underline{M}_k{}^d = 0$, and we have

$$(\bar{\bar{p}}_{2_i} - \bar{\bar{p}}_{1_i})\nabla\alpha_2 = 2\bar{\bar{H}}_{12}\bar{\bar{\sigma}}\nabla\alpha_2. \qquad [69]$$

Thus

$$\bar{\bar{p}}_{2_i} - \bar{\bar{p}}_{1_i} = 2\bar{\bar{H}}_{12}\bar{\bar{\sigma}}, \qquad [70]$$

as expected.

Note from [70], that if the two-phase flow is a bubbly flow with uniform bubbles, then $\bar{\bar{H}}_{21}$ is a constant, and

$$\nabla\bar{\bar{p}}_{2_i} = \nabla\bar{\bar{p}}_{1_i}. \qquad [71]$$

Thus it appears that unequal pressure models do not always give rise to pressure gradient forces. However, there is an effect of surface tension in this model.

The phasic momentum equations become

$$\alpha_d\rho_d\left(\frac{\partial\bar{\underline{v}}_d}{\partial t} + \bar{\underline{v}}_d\cdot\nabla\bar{\underline{v}}_d\right) = -\alpha_d\nabla\bar{\bar{p}}_d + \nabla\cdot[\alpha_d(\bar{\bar{\underline{\underline{\tau}}}}_d + \bar{\bar{\underline{\underline{\tau}}}}_d{}^T)]$$
$$- \underline{M}_c{}^d + 2\bar{\bar{H}}_{cd}\bar{\bar{\sigma}}\nabla\alpha_d + \alpha_d\rho_d\underline{g}_d, \qquad [72]$$

$$\alpha_c\bar{\rho}_c\left(\frac{\partial\bar{\underline{v}}_c}{\partial t} + \bar{\underline{v}}_c\cdot\nabla\bar{\underline{v}}_c\right) = -\alpha_c\nabla\bar{\bar{p}}_c + (\bar{\bar{p}}_c - \bar{\bar{p}}_{c_i})\nabla\alpha_c + \nabla\cdot[\alpha_c(\bar{\bar{\underline{\underline{\tau}}}}_c + \underline{\underline{\tau}}_c{}^T)]$$
$$+ \underline{M}_c{}^d + \alpha_c\rho_c\underline{g}_c, \qquad [73]$$

where $\underline{M}_c{}^d$ is given by [58], $\bar{\bar{p}}_c - \bar{\bar{p}}_{c_i}$ is given by [61], and the turbulent stresses are discussed in section 4.1.

4.4 *A separated flow*

The effect of the constitutive equations are examined for a special *separated* flow. We emphasize that we do not expect the constitutive equations developed for mixed flows to adequately describe separated flow.

Neglect the turbulent stresses, and the virtual mass and lift forces. Assume further that $\bar{v}_k = \bar{v}_k(y)\underline{i}$, $\alpha_k = \alpha_k(y)$, and $g_k = -g\underline{j}$. With these assumptions, the continuity equations are satisfied identically. The momentum equations in the y-direction become

$$0 = -\alpha_k \frac{\partial \bar{\bar{p}}_k}{\partial y} - \alpha_k \bar{\bar{\rho}}_k g . \qquad [74]$$

The momentum equations in the x-direction become

$$0 = -\alpha_k \frac{\partial \bar{\bar{p}}_k}{\partial x} + \frac{\partial}{\partial y} \alpha_k \bar{\bar{\tau}}_{k_{xy}} + M_{k_x} . \qquad [75]$$

From[74], we see that (provided $\bar{\bar{\rho}}_1 \neq \bar{\bar{\rho}}_2$) we have *either*

$$\alpha_1 = 0 ,$$
$$\frac{\partial \bar{\bar{p}}_2}{\partial y} = -\bar{\bar{\rho}}_2 g , \qquad [76]$$

or

$$\alpha_2 = 0 ,$$
$$\frac{\partial \bar{\bar{p}}_1}{\partial y} = -\bar{\bar{\rho}}_1 g . \qquad [77]$$

Thus the flow must indeed be *separated*, with bands or layers of fluid 1 separated by layers of fluid 2. This particular calculation does not show which, if any, of these flows is stable, but it is obvious that in order to avoid Taylor instabilities, the denser fluid must occupy the bottom layer, with the lighter fluid on top. Moreover, if the shear is too great, there exists the possibility of a Kelvin–Helmholtz instability.

This calculation does not pretend to show the evolution to or from separated flow using constitutive equations derived for "mixed" flows. It does suggest, however, that separated flows *can* arise using models derived for mixed flows.

4.5 *Initial and boundary conditions*

To *prove* that the model is well posed, subject to appropriate initial and boundary conditions is beyond the scope of this paper. We shall, however, discuss physically appropriate initial and boundary condition, and argue that the equations are equally as well posed as the Navier–Stokes equation.

As initial conditions, it suffices to prescribe the phasic volume fractions α_k and the velocities $\bar{v}_k$ at $t = 0$. For boundary conditions, it is appropriate to prescribe no-slip conditions on solid walls. At an inlet, we must prescribe $\bar{\bar{p}}_k$, α_k, and some information about relative flow rates, such as $\bar{v}_1 - \bar{v}_2$. The mixture mass flux should then be determined by the flow situations. At outlets, it is necessary to prescribe $\bar{\bar{p}}_k$.

Since the present model contains viscous terms, the partial differential equations are parabolic, and appears to be quite similar to the problem of the flow of two Navier–Stokes fluids. Therefore, this model is well posed in the same sense as the classical single phase fluid flow equations.

5. CONCLUSION

In this paper, we have applied the general principles for formulating constitutive equations to the problem of determining an appropriate multidimensional two-fluid two-phase model. For simplicity, energetic effects were ignored.

In formulating constitutive equations, known forces and effects were accounted for. This leads to constitutive relations which are phenomenologically appealing, and which fit within the general framework for determining constitutive equations.

The equations of motion which we have derived are:

$$\frac{\partial \alpha_d \bar{\bar{\rho}}_d}{\partial t} + \nabla \cdot \alpha_d \bar{\bar{\rho}}_d \bar{\underline{v}}_d = 0, \tag{78}$$

$$\frac{\partial \alpha_c \bar{\bar{\rho}}_c}{\partial t} + \nabla \cdot \alpha_c \bar{\bar{\rho}}_c \bar{\underline{v}}_c = 0, \tag{79}$$

$$\alpha_d \bar{\bar{\rho}}_d \left(\frac{\partial \bar{\underline{v}}_d}{\partial t} + \bar{\underline{v}}_d \cdot \nabla \bar{\underline{v}}_d \right) = -\alpha_d \nabla \bar{\bar{p}}_d + \nabla \cdot [\alpha_d (\bar{\bar{\underline{\tau}}}_d + \underline{\tau}_d{}^T)] - \underline{M}_c{}^d + 2\bar{\bar{H}}_{cd} \bar{\bar{\sigma}} \nabla \alpha_c + \alpha_d \bar{\bar{\rho}}_d \underline{g}_d, \tag{80}$$

$$\alpha_c \bar{\bar{\rho}}_c \left(\frac{\partial \bar{\underline{v}}_c}{\partial t} + \bar{\underline{v}}_c \cdot \nabla \bar{\underline{v}}_c \right) = -\alpha_c \nabla \bar{\bar{p}}_c + (\bar{\bar{p}}_c - \bar{\bar{p}}_{c_i}) \nabla \alpha_c + \nabla \cdot [\alpha_c (\bar{\bar{\underline{\tau}}}_c + \underline{\tau}_c{}^T)] + \underline{M}_d{}^d + \alpha_c \bar{\bar{\rho}}_c \underline{g}_c. \tag{81}$$

Here, $-\underline{M}_c{}^d$ is given by [58], $\bar{\bar{\underline{\tau}}}_k$ is given by [42] and [45], and $\underline{\tau}_k{}^T$ is given by [49].

Let us discuss [78]–[81] in comparison with other models given in the literature. Since most other authors do not attempt to observe invariance requirements, there are some discrepancies between their work, and the present paper. The specific differences occur in the virtual mass terms, where all other authors use a non-invariant form. In addition, the expression for $\underline{\tau}_k{}^T$ used in this paper is more general than that given by Ishii (1975).

The model given by [78]–[81] is still quite complicated, and must be supplemented by experiments which both test the generality of the constitutive equations proposed herein, and evaluate the coefficients in the terms included in these constitutive equations. From the general model, which contains 2771 scalar functions, the specific model of section 4 contains 19 scalar functions which must be determined to specify the model. In addition, most of the known forces in dispersed two phase flow are contained in the above model. These constitutive relations also satisfy all general constitutive requirements except equipresence. This, then, is a model which, while not completely general, should be useful in the analysis of dispersed multidimensional two-phase flow.

The overwhelming number of scalar functions which must be evaluated in the general model preclude its use in any practical situation. In retrospect, this is not totally unexpected, since the general model must be able to handle an extremely large class of flow conditions. The importance of the general constitutive equation approach, therefore, is not in determining a model, but instead, in providing a "filter" for specific terms within a model derived for given flow conditions. The dispersed flow model presented in section 4 provides an excellent example. It is desirable to include a virtual mass force in such a set of equations. The general approach shows that certain forms of the virtual mass acceleration do not satisfy the invariance requirements, and therefore must be ruled out. Thus, all models derived for specific flow situations must fit within the general framework if they are to be considered valid.

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APPENDIX

Consider the process of constructing the objective scalars, vectors and tensors which can appear in the constitutive equations. The scalars can be constructed from vectors and tensors. Furthermore, the determination of the vectors $\underline{V}_i$ and the tensors $\underline{\underline{T}}_j$, needed in [40] and [41] is done in a "bootstrapping" manner, with the tensors used to determine other vectors, and vectors used to determine other tensors.

The natural vectors in [38] are

$$\underline{V}_i = \nabla\alpha_1, \quad \underline{V}_2 = \underline{\bar{v}}_1 - \underline{\bar{v}}_2, \quad \underline{V}_3 = \underline{a}_{12}. \qquad \text{[A1]}$$

The *symmetric* tensors in [38] are

$$\hat{\underline{\underline{T}}}_1 = \underline{\underline{D}}_{1_b}, \quad \hat{\underline{\underline{T}}}_2 = \underline{\underline{D}}_{2_b}. \qquad \text{[A2]}$$

The tensor $\underline{\underline{D}}_{12}$ is not symmetric. Therefore, we write $\underline{\underline{D}}_{12}$ in terms of its symmetric and antisymmetric parts. We have

$$\underline{\underline{D}}_{12} = \frac{1}{2}(\underline{\underline{D}}_{12} + \underline{\underline{D}}_{12}^T) + \frac{1}{2}(\underline{\underline{D}}_{12} - \underline{\underline{D}}_{12}^T). \qquad [A3]$$

But

$$\frac{1}{2}(\underline{\underline{D}}_{12} + \underline{\underline{D}}_{12}^T) = \underline{\underline{D}}_{1_b} - \underline{\underline{D}}_{2_b}.$$

Thus, only the antisymmetric part of $\underline{\underline{D}}_{12}$ is needed in the constitutive equations. For the purposes of this appendix, we shall retain only the antisymmetric part of $\underline{\underline{D}}_{12}$. Moreover, note that an antisymmetric tensor is completely determined from its vector. We shall retain this vector in the list of vectors. Thus, we have

$$\underline{V}_4 = \nabla \times \underline{v}_1 - \nabla \times \underline{v}_2, \qquad [A4]$$

which is the vector constructed from the antisymmetric tensor

$$\frac{1}{2}[\nabla(\underline{\bar{v}}_1 - \underline{\bar{v}}_2) - (\underline{\bar{v}}_1 - \underline{\bar{v}}_2)\nabla].$$

From the tensors in [A2], we can construct more tensors, for use in [41]. The argument used is given by Truesdell & Noll (1965). We have

$$\begin{aligned}
\hat{\underline{\underline{T}}}_3 &= \hat{\underline{\underline{T}}}_1 \cdot \hat{\underline{\underline{T}}}_1,\\
\hat{\underline{\underline{T}}}_4 &= \hat{\underline{\underline{T}}}_2 \cdot \hat{\underline{\underline{T}}}_2,\\
\hat{\underline{\underline{T}}}_5 &= \hat{\underline{\underline{T}}}_1 \cdot \hat{\underline{\underline{T}}}_2 + \hat{\underline{\underline{T}}}_2 \cdot \hat{\underline{\underline{T}}}_1,\\
\hat{\underline{\underline{T}}}_6 &= \hat{\underline{\underline{T}}}_1 \cdot \hat{\underline{\underline{T}}}_1 \cdot \hat{\underline{\underline{T}}}_2 + \hat{\underline{\underline{T}}}_2 \cdot \hat{\underline{\underline{T}}}_1 \cdot \hat{\underline{\underline{T}}}_1,\\
\hat{\underline{\underline{T}}}_7 &= \hat{\underline{\underline{T}}}_2 \cdot \hat{\underline{\underline{T}}}_2 \cdot \hat{\underline{\underline{T}}}_1 + \hat{\underline{\underline{T}}}_1 \cdot \hat{\underline{\underline{T}}}_2 \cdot \hat{\underline{\underline{T}}}_2,\\
\hat{\underline{\underline{T}}}_8 &= \hat{\underline{\underline{T}}}_1 \cdot \hat{\underline{\underline{T}}}_1 \cdot \hat{\underline{\underline{T}}}_2 \cdot \hat{\underline{\underline{T}}}_2 + \hat{\underline{\underline{T}}}_2 \cdot \hat{\underline{\underline{T}}}_2 \cdot \hat{\underline{\underline{T}}}_1 \cdot \hat{\underline{\underline{T}}}_1.
\end{aligned} \qquad [A5]$$

New vectors can be constructed by taking the dot product of the tensors in [A1] and [A5], with the vectors in [A1]. Thus, we have

$$\underline{V}_k = \hat{\underline{\underline{T}}}_i \cdot \underline{V}_j; \quad i = 1,\ldots,8; \quad j = 1,\ldots,4; \quad k = 4,\ldots,36. \qquad [A6]$$

We can now construct symmetric tensors from [A2] and [A6] by forming the symmetric dyads. We have

$$\hat{\underline{\underline{T}}}_k = \underline{V}_i\underline{V}_j + \underline{V}_j\underline{V}_i; \quad i = 1,\ldots,36; \quad j = 1,\ldots,i; \quad k = 9,\ldots,674. \qquad [A7]$$

The process terminates at this level. To see this, consider the possibility of constructing a new vectorial quantity from $\hat{\underline{\underline{T}}}_i$, $k \geq 9$, and $\underline{V}_j$, $j = 1,\ldots,36$. We have

$$\hat{\underline{\underline{T}}}_i \cdot \underline{V}_j = \underline{V}_m(\underline{V}_n \cdot \underline{V}_j) + \underline{V}_n(\underline{V}_m \cdot \underline{V}_j),$$

which is a combination of $\underline{V}_m$ and $\underline{V}_n$. Thus no new vectors can be constructed.

We also note that the process of constructing new tensors, given in [A5], gives no new tensors. To see this, consider

$$\begin{aligned}\hat{\underline{\underline{T}}}_k \cdot (\underline{V}_i\underline{V}_j + \underline{V}_j\underline{V}_i) &= (\hat{\underline{\underline{T}}}_k \cdot \underline{V}_i)\underline{V}_j + (\hat{\underline{\underline{T}}}_k \cdot \underline{V}_j)\underline{V}_i \\ &= \underline{V}_l\underline{V}_j + \underline{V}_m\underline{V}_i\end{aligned}$$

Also,

$$(\underline{V}_i\underline{V}_j + \underline{V}_j\underline{V}_i) \cdot \hat{\underline{\underline{T}}}_k = \underline{V}_i\underline{V}_m + \underline{V}_j\underline{V}_l.$$

Thus the sum of these is

$$(\underline{V}_l\underline{V}_j + \underline{V}_j\underline{V}_l) + (\underline{V}_m\underline{V}_i + \underline{V}_i\underline{V}_m),$$

which is included in [A7].

Consider the scalar invariants which can be constructed from [38]. The natural scalars are

$$S_1 = \alpha_1, \quad S_2 = D_1\alpha_1/Dt, \quad S_3 = D_2\alpha_1/Dt. \qquad [A8]$$

In addition, we note that the scalar product of two vectors is a scalar:

$$\begin{aligned} S_k = \underline{V}_i \cdot \underline{V}_j; \quad & i = 1,\ldots,36; \\ & j = 1,\ldots,i; \\ & k = 4,\ldots,670. \end{aligned} \qquad [A9]$$

In addition, the scalar invariants of the tensors in [A2] and [A5] are

$$\begin{aligned} S_k = \mathrm{tr}(\hat{\underline{\underline{T}}}_j), \quad & j = 1,\ldots,8 \\ & k = 671,\ldots,679. \end{aligned} \qquad [A10]$$

We note that the scalars of the tensors formed by the product of two vectors [A7] are included in the scalars in [A9].

Consider the magnitude of the problem of determining constitutive equations for the variables in [17]. Note that [17] contains four independent tensor functions. Equation [41], together with [A2], [A5] and [A7] give $4 \times 674 = 2696$ scalar functions needed to specify $\bar{\underline{\underline{\tau}}}_k$ and $\underline{\underline{\tau}}_k{}^T$. Each of these 2696 scalar functions can depend on all 679 scalar variables.

There are two independent vector functions in [17]. From [40], and [A1], [A4] and [A6], we see that $2 \times 36 = 72$ scalar functions are needed to specify $\underline{M}_1{}^d$ and $\underline{M}_m$. Again, each of these 72 scalar functions can depend on all 679 scalars.

In addition, there are three scalar functions in [17]. Thus, in all, there are $2696 + 72 + 3 = 2771$ scalar functions needed to specify the constitutive equations in the general model.